§5. Spontaneously broken global symmetries

Consider a symmetry trf. which acts linearly on fields:

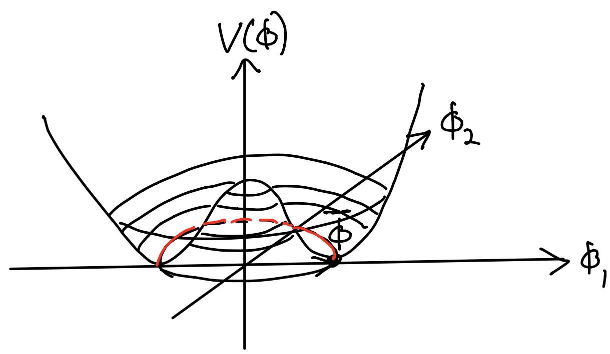
$$\phi_n(x) \longmapsto \phi_n'(x) = \sum_m L_{nm}\,\phi_m(x)$$

→ quantum effective action will have the same symmetry:

$$\Gamma[\phi] = \Gamma[L\phi]$$

For the vacuum, the expectation value of ϕ must be at a minimum of the vacuum energy $-\Gamma[\phi]$, e.g. at $\phi(x) = \bar{\phi}$ (a constant).

If $L\bar{\phi} \neq \bar{\phi}$, we say the "symmetry is broken"!



§5.1 Goldstone Bosons

Specialize to the case of a spontaneously broken "continuous" symmetry (see picture).

$\longrightarrow$ theorem:

the spectrum of physical particles contains one particle of zero mass and spin for *each* broken symmetry

Proof:

suppose the action and measure are invariant under

$$\phi_n(x) \longmapsto \phi_n(x) + i\varepsilon \sum_m t_{nm} \phi_m(x)$$

with $i t_{nm}$ a finite real matrix.

$$\longrightarrow \quad \sum_{n,m} \int \frac{\delta \Gamma[\phi]}{\delta \phi_n(x)} t_{nm} \phi_m(x) \, d^4x = 0 \qquad (1)$$

For translationally invariant theory with constant fields ϕ_n, we have:

$$\Gamma[\phi] = - \mathcal{V} \, V(\phi)$$

where $\mathcal{V}$ is the spacetime volume

→ eq. (1) becomes

$$\sum_{n,m} \frac{\partial V(\phi)}{\partial \phi_n} t_{nm} \phi_m = 0 \qquad (2)$$

Differentiating further with respect to ϕ_ℓ gives:

$$\sum_{n} \frac{\partial V(\phi)}{\partial \phi_n} t_{n\ell} + \sum_{n,m} \frac{\partial^2 V(\phi)}{\partial \phi_n \partial \phi_\ell} t_{nm} \phi_m = 0 \qquad (3)$$

Now specialize to the case where ϕ_n is at a minimum of $V(\phi)$:

$$\sum_{n,m} \left. \frac{\partial^2 V(\phi)}{\partial \phi_n \partial \phi_\ell} \right|_{\phi = \bar{\phi}} t_{nm} \bar{\phi}_m = 0 \qquad (4)$$

Using

$$\frac{\partial^2 V(\phi)}{\partial \phi_n \partial \phi_\ell} = \underset{\substack{\uparrow \\ \text{inverse of propagator}}}{\Delta^{-1}_{n\ell}(0)}$$

$$(4) \longrightarrow \sum_{nm} \Delta^{-1}_{n\ell}(0) t_{nm} \bar{\phi}_m = 0$$

If the symmetry is broken, i.e. $\sum_m t_{nm} \bar{\phi}_m \neq 0$, then eigenvector of Δ^{-1} with zero eigenvalue

→ $\Delta_{n\ell}$ has a pole at $q^2 = 0$

Rank of residue $= \dim \{ t\bar{\Phi} \mid t \in \text{generators} \}$

$\longrightarrow$ one mass-less boson for every independent broken symmetry

"Goldstone boson"

$\square$

<u>Example:</u>

Consider set of N real scalar fields ϕ_n, with Lagrangian

$$\mathcal{L} = -\frac{1}{2} \sum_n \partial_\mu \phi_n \partial^\mu \phi_n - \frac{\mu^2}{2} \sum_n \phi_n \phi_n - \frac{g}{4} \left(\sum_n \phi_n \phi_n \right)^2$$

$\longrightarrow$ invariant under group $O(N)$ rotating ϕ_n

Effective potential for constant fields:

$$V(\phi) \simeq \frac{\mu^2}{2} \sum_n \phi_n \phi_n + \frac{g}{4} \left(\sum_n \phi_n \phi_n \right)^2$$

Suppose g is positive. Then, if μ^2 is positive, minimum of $V(\phi)$ is at $\phi = 0$

$\longrightarrow$ invariant under $O(N)$

If $\mu^2 < 0$, minimum is at $\bar{\Phi}_n$ with

$$\sum_n \bar{\Phi}_n \bar{\Phi}_n = -\mu^2 / g$$

→ mass matrix at tree approximation:

$$M^2_{nm} = \left.\frac{\partial^2 V(\phi)}{\partial\phi_n \, \partial\phi_m}\right|_{\phi=\bar{\phi}}$$

$$= \mu^2 \delta_{nm} + g\, \delta_{nm} \sum_{\ell} \bar{\Phi}_\ell \bar{\Phi}_\ell + 2g \bar{\Phi}_n \bar{\Phi}_m$$

$$= 2g\, \bar{\Phi}_n \bar{\Phi}_m .$$

→ one eigenvector $\bar{\Phi}_n$ with non-zero eigenvalue:

$$m^2 = 2g \sum_n \bar{\Phi}_n \bar{\Phi}_n = 2|\mu^2| ,$$

and $N-1$ eigenvectors perpendicular to $\bar{\Phi}$ with eigenvalue zero.

<u>explanation</u>:

$O(N)$ is broken to $O(N-1)$ (subgroup leaving $\bar{\Phi}$ invariant), so the number of <u>broken</u> symmetries:

$$\frac{1}{2} N(N-1) - \frac{1}{2}(N-1)(N-2) = N-1 \quad \checkmark$$

Another proof for existence of Goldstone bosons:

<u>Proof 2:</u>

any continuous symmetry leads to conserved current: $\dfrac{\partial J^\mu(x)}{\partial x^\mu} = 0 ,$

with a charge $Q = \int d^3x \, iJ^0(\vec{x}, 0)$, satisfying

$$[Q, \phi_n(x)] = -\sum_m t_{nm} \phi_m(x) . \qquad (5)$$

$\longrightarrow$ vacuum expectation value becomes

$$\langle [J^\lambda(y), \phi_n(x)] \rangle_{VAC}$$

$$= (2\pi)^{-3} \int d^4p \left[\rho_n^\lambda(p) e^{ip\cdot(y-x)} - \tilde{\rho}_n^\lambda(p) e^{ip\cdot(x-y)} \right],$$

where we have defined

$$(2\pi)^{-3} i \rho_n^\lambda(p)$$

$$= \sum_N \langle VAC | J^\lambda(0) | N \rangle \langle N | \phi_n(0) | VAC \rangle \, \delta^4(p - p_N),$$

$$(2\pi)^{-3} i \tilde{\rho}_n^\lambda(p)$$

$$= \sum_N \langle VAC | \phi_n(0) | N \rangle \langle N | J^\lambda(0) | VAC \rangle \, \delta^4(p - p_N)$$

$\longrightarrow$ Lorentz invariance gives:

$$\rho_n^\lambda(p) = p^\lambda \rho_n(-p^2) \theta(p^0) ,$$

$$\tilde{\rho}_n^\lambda(p) = p^\lambda \tilde{\rho}_n(-p^2) \theta(p^0) .$$

$$\longrightarrow \langle [J^\lambda(y), \phi_n(x)] \rangle_{VAC} = \frac{\partial}{\partial x_\lambda} \int d\mu^2 \left[\rho_n(\mu^2) \Delta_+(y - x; \mu^2) \right.$$

$$\left. + \tilde{\rho}_n(\mu^2) \Delta_+(x - y; \mu^2) \right)$$

where Δ_+ is the familiar function

$$\Delta_+(z;\mu^2) = (2\pi)^{-3}\int d^4p\, \theta(p^0)\, \delta(p^2+\mu^2)e^{ip\cdot z}$$

For $z^2>0$, Δ_+ depends only on z^2 and μ^2

$\rightarrow$ for $x-y$ spacelike, we have

$$\underbrace{\langle [J^\lambda(y), \phi_n(x)]\rangle_{VAC}}_{\substack{=0 \\ \text{for } x-y \text{ spacelike}}} = \frac{\partial}{\partial y_\lambda}\int d\mu^2\left[\rho_n(\mu^2)+\tilde{\rho}_n(\mu^2)\right]\Delta_+(y-x;\mu^2)$$

$\rightarrow \rho_n(\mu^2) = -\tilde{\rho}_n(\mu^2)$

and thus for general x and y we have

$$\langle [J^\lambda(y), \phi_n(x)]\rangle_{VAC} = \frac{\partial}{\partial y_\lambda}\int d\mu^2 \rho_n(\mu^2)\left[\Delta_+(y-x;\mu^2)-\Delta_+(x-y;\mu^2)\right] \quad (*)$$

Applying the derivative $\frac{\partial}{\partial y^\lambda}$ to both sides of $(*)$ and using $\partial_\lambda J^\lambda = 0$, we get

$$0 = \int d\mu^2\, \rho_n(\mu^2)\,\Box_y\left[\Delta_+(y-x;\mu^2)-\Delta_+(x-y;\mu^2)\right]$$

Using further

$$(\Box_y-\mu^2)\Delta_+(y-x;\mu^2) = 0,$$

we obtain

$$0 = \int d\mu^2\, \mu^2 \rho_n(\mu^2)\left[\Delta_+(y-x;\mu^2)-\Delta_+(x-y;\mu^2)\right]$$

$\longrightarrow \quad \mu^2 \rho_n(\mu^2) = 0$

Setting $\lambda = 0$ in $(*)$ and $x^0 = y^0 = t$, we have

$$\langle [J^0(\vec{y},t), \phi_n(\vec{x},t)] \rangle_{VAC}$$

$$= 2i(2\pi)^{-3} \int d\mu^2 \rho_n(\mu^2) \int d^4p \sqrt{\vec{p}^2+\mu^2} \; e^{i\vec{p}\cdot(\vec{y}-\vec{x})} \delta(p^2+\mu^2)$$

$$= i\delta^3(\vec{y}-\vec{x}) \int d\mu^2 \rho_n(\mu^2)$$

Integrating and using eq. (5), gives

$$-t_{nm} \langle \phi_m \rangle_{VAC} = i \int d\mu^2 \rho_n(\mu^2)$$

$\longrightarrow$ can be only reconciled with $\mu^2 \rho_n(\mu^2) = 0$ iff

$$\rho_n(\mu^2) = i\delta(\mu^2) \sum_m t_{nm} \langle \phi_m(0) \rangle_{VAC} .$$

Using

$$\rho_n^\lambda(p) \sim \sum_N \langle VAC | J^\lambda(0) | N \rangle \langle N | \phi_n(0) | VAC \rangle \, \delta^4(p-p_N)$$

$\longrightarrow$ exists massless state $|N\rangle$ with $\mu^2 = 0$

As $\phi_n(0)|VAC\rangle$ is rotationally invariant

$\longrightarrow$ $\langle N | \phi_n(0) | VAC \rangle$ will vanish for any state state N with non-zero helicity

$\longrightarrow$ massless particle of spin zero!

$\square$